Integrating Analogical and Inductive Learning at Different Levels of Generalization

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Abstract. Several inductive and analogical learning methods have been developed in the past and applied to tasks such as concept formation and classification. However, it seems that the learning of general principles and abstract coherences can hardly be achieved by these methods when taken separately. In this paper, we argue that inductive and analogical learning address complementary tasks and therefore can be combined in a prolific manner. We present a three-level stage model in which analogical and inductive elements are used to generalize and integrate knowledge from different domains. Via analogy, commonalities of two different domains are identified, and thereby hints for abstract concepts are provided. Inductive methods are used to refine these concepts - and by transfer into new domains they are further elaborated. Our approach is applied to demonstrate how general physical principles can be learned from a series of examples from astronomy, nuclear physics, classical mechanics, and electrodynamics.

1 Introduction

In psychology, cognitive science, and artificial intelligence, learning has been discussed in many different facets and diverse learning strategies have been proposed. However, none of these approaches can explain how humans extract abstract principles across various domains. For example, general physical principles such as the “equilibrium of forces” or complex systems such as a “central force system” can be discovered in many different domains, but cannot be generalized via simple induction, because the concrete examples in which these general principles occur differ due to context and domain differences (e.g. gravitational force systems in astronomy or electromagnetic force systems have not much in common except the basic principle of a general force system).

This paper proposes an approach explaining how general physical principles can be learned and understood. We propose a learning-by-levels strategy that combines two learning mechanisms: One is analogical learning which is used to compare the (sometimes rather) different domains. The following example illustrates how different such domains can be: Consider the gravity force which originates from the sun due to its mass, the tractive force that a chain conveys to the seats of a merry-go-round or the coulomb force between nucleus and electron. These are all attracting forces, but have completely different reasons. The other learning mechanism is inductive learning which can generalize over a large amount of data samples and propose abstract generalizations such as general principles.

The remainder of this the paper is structured as follows: The second section shortly explains the differences between analogical and inductive learning (from a cognitive science perspective). Afterwards we introduce heuristic-driven theory projection (HDTP), our model for computing analogies. Section four describes the learning-by-levels strategy and explains how analogical and inductive learning interact. In section 5 we outline the learning process step-by-step with several examples.

2 Induction and Analogies as Different Approaches to Learning

Inductive learning and analogical learning are the constituents on which our learning-by-levels approach is built.

Inductive Learning creates knowledge via generalization over a large set of data samples which share common properties, facts or laws and deduces some general principles. The likeliness of inductive inferences increases with the number of valid cases. Our inductive learning strategy differs in some respect from the classical term "induction" as used in AI and machine learning: While induction requires a (rather) large set of data samples to draw sound conclusions, humans apply the inductive strategy already with a limited number of examples (e.g. four or five cases). Inductive learning is typically applied in one of the following three situations. *(1) Quantitative refinement:* induction refines the conceptualization of a domain. From the observation that water freezes in an environment of 0°C, -10°C and -20°C, we conclude that it freezes for all temperatures less than 0°C. Based on the refinement we make a general statement for a set of cases or intervals. *(2) Recognition of qualitative regularities*: Induction is used to find regularities across a set of data samples and to identify a general law which holds across all data samples. *(3) Category learning and concept formation*: Induction is the main driver for category learning or concept formation. Once a common pattern is recognized across a number of samples, these samples are grouped together and constitute examples for one category [1].

Analogical Learning differs from inductive learning: It typically requires only two domains - a source and a target domain. Analogies aim to identify structural commonalities, analogous relations or objects playing the same roles [2]. These commonalities are captured in an abstract generalization (section 3), however, this generalization is not assumed to be generally true, but only for the specific source and target domain. Analogical learning is typically applied across different domains. The purpose of analogies is to adapt knowledge available about the source such that it can be applied to the target in a way that new analogous inferences can be drawn. Learning based on analogy have been discussed extensively in literature. For instance, Klenk and Forbus apply Gentner's Structure Mapping Engine [3, 4] to solve qualitative reasoning problems in physics by analogy [5]. This approach is weakly related to our approach, however, Klenk and Forbus only focus on the first level of analogical learning (compare figure 2 for the different stages). In this paper, we will restrict our considerations to the analogy model HDTP [6], which is a formal-logic approach for analogy-making. Although other approaches in machine learning such as transfer learning or multi-level learning [7, 8] have similarities in spirit with our approach to analogical learning, they use fundamentally different techniques (e.g. probabilistic or neural) and are incompatible with our logical approach.

Although analogical learning and inductive learning differ, they do not contradict to each other and can be applied within one overall learning strategy. Inductive learning can generalize commonalities, but it cannot detect commonalities across different domains which are not directly comparable. This is one of the strengths of analogical reasoning. On the other hand, analogical reasoning compares only two domains and therefore cannot propose general laws across many domains that might lead to general physical principles. Inductive learning is required for this.

3 Analogical Learning with Heuristic-Driven Theory Projection

Heuristic-Driven Theory Projection (HDTP) is a formally sound theory for computing analogical relations between a source and a target domain. HDTP computes analogies not only by associating concepts, relations, and objects, but also complex rules and facts between target and source domain. We refer to [6] for the specification of the syntactic, semantic, and algorithmic properties of HDTP.

Syntactically, HDTP is defined on the basis of a many-sorted first-order language. First-order logic is used in order to guarantee the necessary expressive power of the theory. An important assumption is that analogical reasoning crucially contains a generalization (or abstraction) process. In other words, the identification of common properties or relations is represented by a generalization of the input of source and target. Mathematically this can be modeled by an extension of the theory of anti-unification [9, 10], a mathematically sound account describing the possibility of generalizing terms of a given language using substitutions. More precisely, an anti-unification of two terms t_1 and t_2 can be interpreted as finding a generalized term t (or structural description t) of t_1 and t_2 which may contain variables, together with two substitutions θ_1 and θ_2 of variables, such that $t\theta_1 = t_1$ and $t\theta_2 = t_2$. Because there are usually many possible generalizations, anti-unification tries to find the most specific one (cp. also [11]).

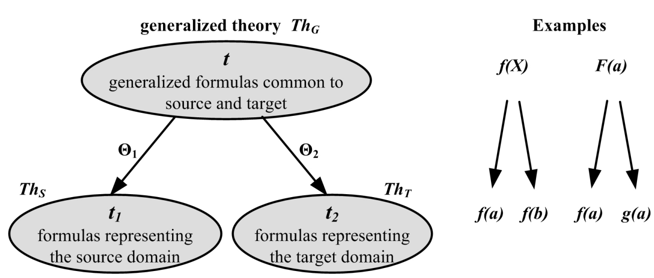

Fig. 1. Establishing the analogical relation between the source theory Th_S and the target theory Th_T and constructing the general theory Th_G.

Given two theories Th_S and Th_T modeling source and target domain as input, the HDTP algorithm computes the analogy. Table 1 shows the algorithm, which contains the following steps:

Input: A theory Th_S of the source domain and a theory Th_T of the target domain represented in a many-sorted predicate logic language.

Output: A generalized theory Th_G such that the input theories Th_S and Th_T can be reestablished by substitutions.

Algorithm: Selection and generalization of facts and rules. Select an axiom from Th_T according to a heuristics h. In HDTP, this heuristics could select formulas according to their complexity, i.e. prefer less complex literals before complex rules. Afterwards, select an axiom from Th_S and construct a generalization (together with corresponding substitutions).

Optimize the generalization w.r.t. a given heuristics and update Th_G w.r.t. the result of this process. The heuristics used by HDTP orders the anti-instances according to the complexity of their substitutions (e.g. length of substitutions).

Transfer (project) facts and laws of Th_S to Th_T provided they are not generalized yet. Test (using an oracle) whether the transfer is consistent with Th_T. This can be done via experiments or using world knowledge in a database.

```
T = axioms of the target domain sorted by a heuristics h
S = axioms of the source domain
G = empty list of axioms of generalized theory
Θ1 = Θ2 = empty substitution
Th_T^{Ah} = Th_T
FOR φ ∈ T
    T = normal_form(T)
    SELECT ψ ∈ S
        ψ = normal_form(ψ)
        IF not same_structure(φ, ψ) REJECT
        SELECT (ξ, Θ1, Θ2) ∈ anti_instances(φ, ψ, Θ1, Θ2)
    WITH ξ best according to a heuristics h'
    IF h'(ξ) > a given threshold
        ADD ξ to G
        ADD ξΘ2 to T_T^{Ah}
        REMOVE ψ from S
    ELSE FAIL
END FOR
FOR ψ ∈ S
    φ = transfer(ψ, Θ1, Θ2)
    IF T_T^{Ah} ⊢ ¬φ CONTINUE
    IF oracle(φ) = FALSE CONTINUE
    ADD φ to Th_T^{Ah}
    ADD generalize(ψ, Θ1) to G
END FOR
```

Table 1. The algorithm HDTP-A generalizing two theories Th_S and Th_T.

For our purpose it is important to stress the two ways of analogical learning: *(1) Learning by analogical transfer* refers to new knowledge gained in the target domain. Usually, the source domain is the more familiar domain described by a richer theory. By drawing the analogy, HDTP transfers knowledge of the source domain to the target domain. *(2) Learning by generalization* is a by-product of the analogy-making process: Via the theory of anti-unification, HDTP computes a generalized theory which contains the commonalities between Th_S to Th_T. The "learning by generalization" is the important step for our proposed stage model.

4 Learning by Levels

None of the above described learning theories can explain the whole process of learning and understanding general principles across different domains. Analogical reasoning can compare different domains by analyzing their common structure, but the analogical generalizations hold only for the specific source and target domain. On the other hand, inductive learning seems to be inadequate, because it can find regularities only from data samples containing the same (type of) information. This means, that induction cannot identify commonalities across different domains or contexts, because these often differ in their superficial properties and commonalities exist only with respect to an analogous structure. Inductive learning requires a large number of examples to supply enough evidence for the generalization. Unlike analogical reasoning, we consider inductive generalizations as universally quantified laws. Therefore we suggest to enhance analogical learning with inductive mechanisms and propose three levels of learning.

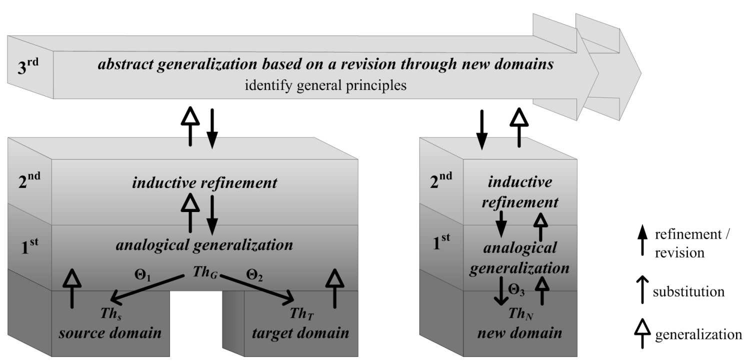

Fig. 2. A stage model for learning by levels: Integrating analogical and inductive learning.

Figure 2 illustrates the stage model of the learning process. The basis for the learning process are the domain theories. In the case of analogical learning these are a source domain Th_S and a target domain Th_T (left side of the figure). The three levels (see also [12]) contribute to the learning process as follows:

1. Level: At the lowest level of learning the source and the target domain are compared via analogy to identify common structures between the different domains. The commonalities are generalized in a general theory Th_G. It is important to stress that the knowledge contained in Th_G is only general in so far as it applies to the source and target domain, but not generally across many or even all domains. Commonalities identified at this learning stage are existentially quantified within the two domains, i.e. there exists some parameter configuration for which the analogical generalization holds.

2. Level: At this level of learning the analogy proposed at the 1. level has to be tested, because the analogical knowledge transfer might be true only for prototypical situations. Therefore the knowledge transfer must be tested for various cases in the target domain. The process runs through a number of different refinement steps:

- To investigate the range parameters for which the analogy holds, new knowledge inferred by the analogy (the analogical transfer) is tested with experiments for a large number of different parameters. Via induction we can infer certain intervals for which the analogy holds.
- If the analogy was shown to be wrong in some cases, the parameter configuration is selected for which the analogy no longer holds.
- The results are projected back to the source, to test experimentally whether the theory holds for this parameter configuration. If necessary, the source domain theory is adjusted.
- New findings in the source domain are included in the analogy and projected back to the target domain.

In section 5, we explain the analogy between the relation of the moon to the earth and the solar system. We show that an abstract generalization of a central force system holds for the earth-moon relation, the solar system, and for new domains such as the rotating seats of a merry-go-round. In all three domains exists an orbiting element and an attracting force. The inductive refinement compares possible values for the intensity of the attracting forces and the velocity: if the velocity is too small or too fast, the rotating object leaves the orbit. However, the possible parameter settings might differ in the solar system and the merry-go-round.

The outcome is a refined analogy. This level is called inductive refinement, because it applies induction for the parameter check. While the generalized facts and laws on the first level were existentially quantified (a parameter-configuration exists for which the analogical generalization holds), the refinement at the second level can be considered to be universally quantified, but still restricted to Th_S and Th_T.

3. Level: The aim of this level is the identification of general principles - in our example these are general principles in physics such as the equilibrium of forces or general systems such as the central force system. This learning step goes beyond the comparison of two domains and requires inductive learning (in the sense of section 2) on top of analogical comparison. At this level, the learning process starts with an intuitive hypothesis about a general principle and compares this iteratively with other domains to gain more confidence. This process leads to a revision of the general principle, which in the end contains only those elements important for the principle and no domain-specific attributes. Such general principles are considered to be universally quantified across different domains.

The three levels proposed above outline the different mechanisms applied (pure analogical learning or learning based on inductive strategies). However, we do not claim that humans learn sequentially in such levels. The identification of an abstract generalization is by no means a straightforward process. It is driven by intuitive hypotheses of possible generalizations and search for new domains supporting this hypothesis. If the new domain does not support the hypothesis, the abstract generalization either does not apply to this domain or it can be revised. A revision takes place, if a meaningful generalization can be found, which applies to the old and the new domains. This iterative process leads to the elaboration of a general principle. Learning is a process in which different learning mechanisms interact and are iteratively repeated to refine and revise knowledge on the various abstraction levels. We present a model explaining how to learn general principles based on the analogical generalization produced by HDTP.

5 Illustration with Examples

This section explains the process of learning the general physical principle "central force system" using various examples. We use HDTP to compute analogical generalizations and explain how the abstract generalization can be reached.

5.1 Analogical Generalization

Our first example compares the relation between the earth and its moon with the solar system (i.e. the relation between sun and earth). The observation that the earth has a gravitational effect and attracts objects and that the moon revolves around the earth in a more or less circular motion leads to the following conceptualization (table 2).

Table 2. The formalization of the relation between earth and moon and of the solar system.

Formalization of the earth-moon relation (Th_S)	Formalization of the sun-earth relation (Th_T)
types	**types**
$real, object, time$	$real, object, time$
constants	**constants**
$earth : object, moon : object$	$sun : object, earth : object$
functions	**functions**
$mass : object \rightarrow real \times \{kg\}$ $dist : object \times objct \times time \rightarrow real \times \{m\}$ $gravity : object \times object \times time \rightarrow real \times \{N\}$ $centrif : object \times object \times time \rightarrow real \times \{N\}$	$mass : object \rightarrow real \times \{kg\}$ $dist : object \times object \times time \rightarrow real \times \{m\}$ $gravity : object \times object \times time \rightarrow real \times \{N\}$ $centrif : object \times object \times time \rightarrow real \times \{N\}$
facts	**facts**
$mass(earth) > mass(moon) > 0$ $\forall t : time : gravity(moon, earth, t) > 0$ $\forall t : time : dist(moon, earth, t) > 0$	$mass(sun) > mass(earth) > 0$ $\forall t : time : gravity(earth, sun, t) > 0$ $\forall t : time : dist(earth, sun, t) > 0$
laws	**laws**
1. $\forall t : time, o_1 : object, o_2 : object :$ $mass(o_1) > 0 \wedge mass(o_2) > 0$ $\rightarrow gravity(o_1, o_2, t) > 0$	1. $\forall t : time, o_1 : object, o_2 : object :$ $mass(o_1) > 0 \wedge mass(o_2) > 0$ $\rightarrow gravity(o_1, o_2, t) > 0$
2. $\forall t : time, o_1 : object, o_2 : object :$ $dist(o_1, o_2, t) > 0 \wedge gravity(o_1, o_2, t) > 0$ $\rightarrow centrif(o_1, o_2, t) = -gravity(o_1, o_2, t)$	2. $\forall t : time, o_1 : object, o_2 : object :$ $dist(o_1, o_2, t) > 0 \wedge gravity(o_1, o_2, t) > 0$ $\rightarrow centrif(o_1, o_2, t) = -gravity(o_1, o_2, t)$
3. $\forall t : time, o_1 : object, o_2 : object :$ $0 < mass(o_1) < mass(o_2) \wedge$ $dist(o_1, o_2, t) > 0 \wedge centrif(o_1, o_2, t) < 0$ $\rightarrow revolves\_around(o_1, o_2)$	3. $\forall t : time, o_1 : object, o_2 : object :$ $0 < mass(o_1) < mass(o_2) \wedge$ $dist(o_1, o_2, t) > 0 \wedge centrif(o_1, o_2, t) < 0$ $\rightarrow revolves\_around(o_1, o_2)$

The gravitational effect of a heavenly body depends on its mass. Due to the earth's mass there exists terrestrial gravity which attracts the moon. Moreover, we can observe that earth and moon have a positive (more or less constant) distance over time and the moon does not "fall down" to the earth. If the terrestrial gravity attracts the moon, but it stays distant from the earth, there must exist another force - a counter force to the gravitational force - which is called centrifugal force (abbreviated by $centrif$ in the formalization). From the facts that moon and earth have a positive distance, the earth's mass is greater than that of its moon and there exists some centrifugal force we conclude that the moon revolves around the earth.

The rotational relationship between the earth and its moon is actually analogous to the relationship between the sun and the earth. The analogy model HDTP computes the analogical relation via comparing the source theory describing earth/moon to the target theory describing sun/earth and computes a generalized theory (table 3).

Table 3. The analogical generalization of the earth/moon relation and the sun/earth relation.

types
$real, object, time$
constants
$X : object, Y : object$
functions
$mass : object \to real \times \{kg\}$
$dist : object \times object \times time \to real \times \{m\}$
$gravity : object \times object \times time \to real \times \{N\}$
$centrif : bject \times object \times time \to real \times \{N\}$
facts
$mass(X) > mass(Y) > 0$
$\forall t : time : gravity(X, Y, t) > 0$
$\forall t : time : dist(X, Y, t) > 0$
laws
1. $\forall t : time, o_1 : object, o_2 : object :$
$mass(o_1) > 0 \wedge mass(o_2) > 0$
$\to gravity(o_1, o_2, t) > 0$
2. $\forall t : time, o_1 : object, o_2 : object :$
$dist(o_1, o_2, t) > 0 \wedge gravity(o_1, o_2, t) > 0$
$\to centrif(o_1, o_2, t) = -gravity(o_1, o_2, t)$
3. $\forall t : time, o_1 : object, o_2 : object :$
$0 < mass(o_1) < mass(o_2) \wedge$
$dist(o_1, o_2, t) > 0 \wedge centrif(o_1, o_2, t) < 0$
$\to revolves\_around(o_1, o_2)$

As analogies refer to structural commonalities, HDTP analyzes common relations and common roles and detects, that earth$_{Th_S}$ and sun play the same role: they are the objects with the greater mass, they have a positive distance to the orbiting objects and have a gravitational effect on them. Moon and earth$_{Th_T}$ play the same role as the orbiting objects. Also the laws are generalized. It is important to stress that analogy is required to find such role commonalities: a superficial search for commonalities would have aligned earth$_{Th_S}$ with earth$_{Th_T}$.

5.2 Abstract Generalization

The relationship between earth/moon or sun/earth is not unique in our universe. Actually, this relation holds for all heavenly bodies. Based on our analogy (section 5.1) humans are likely to come up with a hypothesis about a universal law of gravitation or an even more abstract law about the relation of forces and a resulting rotation. At the level of abstract generalization, the learner with the above described hypothesis searches for other domains to support or reject his hypothesis. We distinguish two effects: On the one hand, this is a refinement process of the abstract generalization. New domains either support the hypothesis and increase the likeliness of the abstract generalization or they do not support the hypothesis. If only parts of the hypothesis are rejected, it can be revised. If the hypothesis as a whole is rejected, this can mean that the new domain was no example for the abstract generalization. On the other hand, the hypothesis of an abstract generalization also determines the way the learner conceptualizes the new domain. The following description of the Rutherford Atom is an example for this case.

The central body system. The analogical generalization describes a gravitation system with one center of gravitation, one orbiting body and two forces: the gravitational force is an attracting force and the centrifugal force is its counter force. The conceptualization of the gravitation system influenced mainly Rutherford's construction of the atom model. Based on the idea of a gravitation system Rutherford developed his atom model with electrons revolving around the nucleus (table 4, left side).

Table 4. The hypothesis, that the gravitation system is a general principle, guides Rutherford's conceptualization of an atom (left side) and leads to a revised general principle: the "central body system" (right side).

Formalization of the Rutherford Atom	Abstract generalization: Central body system
types	**types**
$real, object, time$	$real, object, time$
constants	**constants**
$nucleus : object, electron : object$	$X : object, Y : object$
functions	**functions**
$mass : object \to real \times \{kg\}$	$mass : object \to real \times \{kg\}$
$dist : object \times object \times time \to real \times \{m\}$	$dist : object \times object \times time \to real \times \{m\}$
$electric\_charge : object \to real \times \{eV\}$	$F : object \times object \times time \to real \times \{N\}$
$coulomb : object \times object \times time \to real \times \{N\}$	$centrif : object \times object \times time \to real \times \{N\}$
$centrif : object \times object \times time \to real \times \{N\}$	
facts	**facts**
$mass(nucleus) > mass(electron) > 0$	$mass(X) > mass(Y) > 0$
$electric\_charge(electron) < 0$	$\forall t : time : F(X, Y, t) > 0$
$electric\_charge(nucleus) > 0$	$\forall t : time : dist(X, Y, t) > 0$
$\forall t : time : coulomb(electron, nucleus, t) > 0$	
$\forall t : time : dist(electron, nucleus, t) > 0$	
laws	**laws**
1. $\forall t : time, o_1 : object, o_2 : object :$ $electric\_charge(o_1) > 0$ $\wedge electric\_charge(o_2) < 0$ $\to coulomb(o_1, o_2, t) > 0$	1. $\forall t : time, o_1 : object, o_2 : object :$ $Feature1(o_1) \wedge Feature2(o_2)$ $\to F(o_1, o_2, t) > 0$
2. $\forall t : time, o_1 : object, o_2 : object :$ $dist(o_1, o_2, t) > 0 \wedge coulomb(o_1, o_2, t) > 0$ $\to centrif(o_1, o_2, t) = -coulomb(o_1, o_2, t)$	2. $\forall t : time, o_1 : object, o_2 : object :$ $dist(o_1, o_2, t) > 0 \wedge F(o_1, o_2, t) > 0$ $\to centrif(o_1, o_2, t) = -F(o_1, o_2, t)$
3. $\forall t : time, o_1 : object, o_2 : object :$ $0 < mass(o_1) < mass(o_2) \wedge dist(o_1, o_2, t) > 0 \wedge$ $centrif(o_1, o_2, t) < 0$ $\to revolves\_around(o_1, o_2)$	3. $\forall t : time, o_1 : object, o_2 : object :$ $0 < mass(o_1) < mass(o_2) \wedge$ $dist(o_1, o_2, t) > 0 \wedge F(o_1, o_2, t) < 0$ $\to revolves\_around(o_1, o_2)$

Many analogous commonalities exist here: There are two types of objects, a nucleus and electrons. The nucleus has a greater mass than the electrons. They attract each other, however the attracting coulomb force is caused by the opposite electric charges of electron and nucleus. The general principle of the "gravitation system" can be applied to the atom using the mechanism of analogy and leads to the revised general principle "central

Table 5. Formalization of the merry-go-round and the electron in a homogeneous magnetic field.

Formalization of the merry-go-round	Electron in a homogeneous magnetic field
types	**types**
$real, object, location, time$	$real, object, location, time$
constants	**constants**
$seat : object, center : location$	$electron : object, center : location$
functions	**functions**
$mass : object \rightarrow real \times \{kg\}$	$mass : object \rightarrow real \times \{kg\}$
$dist : object \times location \times time \rightarrow real \times \{m\}$	$dist : object \times location \times time \rightarrow real \times \{m\}$
$tractivef : object \times object \times time \rightarrow real \times \{N\}$	$lorentzf : object \times object \times time \rightarrow real \times \{N\}$
$centrif : object \times location \times time \rightarrow real \times \{N\}$	$centrif : object \times location \times time \rightarrow real \times \{N\}$
facts	**facts**
$mass(seat) > 0$	$mass(electron) > 0$
$\forall t : time : revolves\_around(seat, center, t)$	$\forall t : time : revolves\_around(electron, center, t)$
laws	**laws**
1. $\forall t : time, o : object, l : location :$	1. $\forall t : time, o : object, l : location :$
$mass(o) > 0 \wedge dist(o, l, t) > 0 \wedge$	$mass(o) > 0 \wedge dist(o, l, t) > 0 \wedge$
$revolves\_around(o, l, t)$	$revolves\_around(o, l, t)$
$\rightarrow centrif(o, l, t) < 0$	$\rightarrow centrif(o, l, t) < 0$
2. $\forall t : time, o : object, l : location :$	2. $\forall t : time, o : object, l : location :$
$dist(o, l, t) > 0 \wedge centrif(o, l, t) < 0$	$dist(o, l, t) > 0 \wedge centrif(o, l, t) < 0$
$\rightarrow tractivef(o, l, t) = -centrif(o, l, t)$	$\rightarrow lorentzf(o, l, t) = -centrif(o, l, t)$

body system" (table 4, right side). The atom domain supports the hypothesis that the gravitation system could be generalized across other domains and leads to a revised, more general concept which is not restricted to heavenly bodies and gravitational force.

The central force system. The revision process of the abstract generalization is continued by comparing the "central body system" with two other domains: a merry-go-round and an electron in a homogeneous magnetic field.

The merry-go-round (left of table 5) is a rotating system, but it differs from the previous examples with regard to the center: there is no *body* in the center of the circular motion. The seats are the revolving objects: the centrifugal force pushes them outside, away from the center. A tractive force (called $tractivef$ in the formalization) is the attracting counter force and acts on the seats via the metal chains connecting the seats with the center of the merry-go-round.

The conceptualization of an electron in a homogeneous magnetic field (right of table 5) is also a rotational system: It can be observed that an electron in a magnetic field moves on a circular motion which has a center point. Therefore a centrifugal force acts on the electron. Since it moves continuously on the same orbit, a counter-force towards the center of the orbit must exist (called $lorentzf$). The examples merry-go-round and the electron in the magnetic field revise the abstract generalization of a "central force system" with regard to the central body. Now the learner recognizes that it is enough to have a central location of the orbit which does not need to be an object.

The General Principle. The general principle resulting from the revision process is a "central force system" explaining the relationship between an attracting force towards a center, a centrifugal force and an object revolving around the center.

Table 6 shows the formalization of the resulting abstract generalization. It contains only the information which is necessary for a general force system and reduces domain specific knowledge such as mass differences between bodies in the gravitation system or electric charges. Also it is irrelevant whether one observes the forces and the positive distance first or the revolving motion of the orbiting object. Figure 3 illustrates

Table 6. The abstract generalization of the general principle.

types
$real, object, location, time$
constants
$O : object, L : location$
functions
$mass : object \rightarrow real \times \{kg\}$
$dist : object \times location \times time \rightarrow real \times \{m\}$
$F : object \times location \times time \rightarrow real \times \{N\}$
facts
$mass(O) > 0$
$\forall t : time : F(O, L, t) \neq 0$
$\forall t : time : dist(O, L, t) > 0$
laws
1. $\forall t : time, o : object, l : location :$
$dist(o, l, t) > 0 \wedge F(o, l, t) \neq 0$
$\rightarrow \exists counterforce : counterforce(o, l, t) = -F(o, l, t) \wedge$
2. $\forall t : time, o : object, l : location :$
$dist(o, l, t) > 0 \wedge mass(o, t) > 0$
$\rightarrow (centrif(o, l, t) < 0 \leftrightarrow revolves\_around(o, l))$

the line of argument to understand the revision process of the abstract generalization. The earth/moon and the sun/earth relation as well as the Rutherford atom - all examples for the central body system - start from the discovery of the attracting force and conclude from the positive distance that there must exist a centrifugal force which induces a revolving motion. The merry-go-round and the electron in the homogeneous magnetic field start with the observation of a revolving object with a centrifugal force and conclude that there also must exist an attracting force towards the center of the orbit.

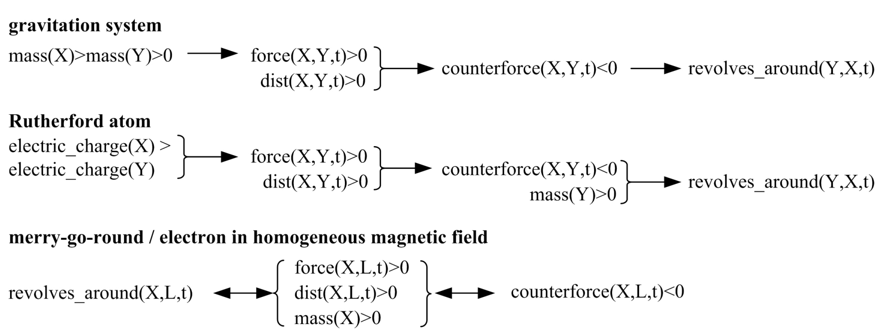

Fig. 3. The line of argumentation for the various examples leading to the abstract generalization

6 Conclusions and Future Work

Analogical reasoning is a fundamental construct of learning, because it is essential to compare commonalities across different domains. However, to reach abstract generalizations such as general physical principles we need to enhance analogies with an inductive aspect. This leads to the establishment of generalizations which apply across various domains. This paper outlined three levels of learning and illustrated the step from the analogical generalization to the abstract generalization with a chain of example. The examples chosen in this paper lead to the general principle of a central force system and an abstract idea of force. The establishment of even more general principles of physics such as the equilibrium of forces or general laws of nature such as the conservation of energy follows the same learning strategy, but the generalization and revision process of the abstract generalization is driven further.

In a future application scenario, an e-learning tutor might use our stage model of learning to explain one domain via another domain which is easier to conceptualize and understand by the student. Analogous domains could be detected autonomously and related to a general principle. This teaching strategy is often used by human teachers. Future work will address the further development of a language for a formal specification of the learning stages and an algorithm for the abstract generalization and the revision process.

Acknowledgement
The work was supported by the German Research Foundation (DFG) through the project “Modeling of predictive analogies by Heuristic-Driven Theory Projection” (grant KU 1949/2-1).

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