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Forward and reverse waves in the one-dimensional model of the cochlea

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Consideration of a source of oto-acoustic emission in a cochlear model implies consideration of the types of waves that such a source can emit. One wave travels in the normal, forward, direction. As any other forward wave it undergoes little or no reflection and it eventually disappears completely because of dissipation. The other wave travels in the reverse direction and it appears to undergo appreciable reflection. In the present paper this phenomenon is studied via the use of two appropriately simplified long-wave models of the cochlea. One model, the exponential model, puts emphasis on the variation of the stiffness along the length of the basilar membrane. The second model concentrates on what happens in the region of resonance. The latter model turns out to have the largest predictive power for the problem at hand. Consideration of the flow of energy in the cochlear fluid brings forth the explanation why in the used model of the cochlea reflection conditions at the stapes have such a surprisingly small influence on the operating conditions of a potential source of emission.

cochlear mechanics, basilar membrane, impedance, wave propagation, long-wave cochlear model

Introduction

A source of oto-acoustic emission is usually considered to be situated at a particular location along the length of the basilar membrane (BM). Waves will emerge from such a source in two directions: the forward and the reverse direction, i.e. from stapes to helicotrema and from helicotrema to stapes, respectively. On theoretical grounds it has been predicted that cochlear models will support these two types of waves quite differently (De Boer and Viergever, 1984). Forward waves will propagate almost without reflection but for reverse waves this is not the case. For the one-dimensional model the prediction has been verified (De Boer et al., 1985; Kaernbach et al., 1980), the physical interpretation of this property, however, is far from complete, to say the least. It is the purpose of the present paper to shed more light on the properties of forward and reverse waves with the aim to obtain more insight into the phenomenon of oto-acoustic emission.

As in the last two papers mentioned we will use the passive one-dimensional model of the cochlea and its transmission-line analogue. By using analytical solutions to simplified forms of the model we simulate forward and reverse waves in more complex models and can then try to understand why forward and reverse waves are so different. There is a related problem: the physical conditions under which a sound emitter operates are found to depend surprisingly little upon reflection of reverse waves by the stapes (De Boer et al., 1985). This problem can be attacked with the same technique. It is then shown, via consideration of the energy flow, why reverse waves, when they are reflected, reach their source with such a reduced amplitude that they hardly influence the operating conditions of that source.

To facilitate the organization of this paper we will designate the various models to be studied by letters. Model 'M' is the 'complete' model, the BM impedance $\zeta(x) - x$ denotes the location along the BM – has stiffness, resistance and mass components, the last one is constant and the former two are exponential functions of x. In this way the fact that the stiffness of the BM varies over a very large range is represented. Due to the presence of

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a mass component the BM shows resonance at a location that is specific for each frequency. The impedance function is given by:

$$\zeta(x) = C_0 \exp(-\alpha x)/i\omega + R_0 \exp(-\alpha x/2) + i\omega M_0, \quad (1a)$$

$$\text{with } R_0 = \delta(C_0 M_0)^{1/2}. \quad (1b)$$

C_0 is the stiffness coefficient at $x = 0$ (the stapes location), α is the stiffness space constant, ω is the radian frequency, M_0 is the mass constant and δ the damping fraction. The resistance function is chosen in such a way that, for any frequency, the resistance is δ times the mass reactance ωM_0 at the location where resonance obtains for that frequency.

Model 'E' is the 'exponential model', the BM impedance contains only the first term of Eqn. 1a. In this model the emphasis is put upon the way the stiffness varies with x and the consequences of that behaviour on the response. Originally, this model is lossless. By way of a simple transformation the model is made to acquire non-zero damping. Model 'R' is the 'resonance model', designed to simulate conditions around the resonance location. In this model the BM impedance $\zeta(x)$ is assumed to be a linear function of x in such a way that the imaginary part of $\zeta(x)$ goes through zero at a certain location $x = x_0$ (this location is the resonance location for the frequency under study).

Model E: exponential model

The exponential model was the basis of the early pioneering work of Zwislocki (1948) who could explain with this model many of the then known experimental facts about the cochlea. In later years the model has been expounded upon by Dallos (1973), Schroeder (1973), Zweig et al. (1976) and De Boer (1980). For the present paper the model will be modified slightly to incorporate losses. This is done to simulate more realistic situations. The BM impedance in model E is the first term of Eqn. 1a but the variable x is replaced by ξ which, in its turn, is made equal to $x - i\delta/\alpha$. At any location x along the BM the ratio of resistance to reactance then is the same, namely $-\tan\delta$ (resistance positive, reactance negative).

For this case the solution for the pressure $p(x)$ is a linear combination of the Bessel functions $J_0(z)$ and $Y_0(z)$:

$$p(x) = p_0\{J_0(z) + \gamma Y_0(z)\}, \quad (2)$$

where z is an exponential function of ξ:

$$z = D_1 \exp(\alpha\xi/2), \quad (3)$$

with the constant D_1 given by

$$D_1 = \frac{2\omega}{\alpha}\left(\frac{2\rho}{hC_0}\right)^{1/2}. \quad (4)$$

The parameter h is the (effective) channel height and ρ the fluid density. When the model is driven from the left, i.e. from the stapes location $x = 0$, the solution is Eqn. 2 with $\gamma = -i$ (De Boer, 1980). This solution represents forward waves that (mainly) travel to the right, in the direction of increasing x, and suffer very little reflection.

In the present case we assume the model to be driven from the right, i.e. from a location $x = x_d$ anywhere along the BM. At a location $x = x_a$ to the left of the driving point the model is assumed to be loaded with the impedance Z_a. If we denote by z_a and ξ_a the values of z and ξ corresponding to $x = x_a$, respectively, the value of γ is found to be

$$\gamma = \frac{-iJ_0(z_a) - Z_a D_a J_1(z_a)}{iY_0(z_a) + Z_a D_a Y_1(z_a)}, \quad (5)$$

where D_a equals $(2/\rho h C_0)^{1/2} \exp(\alpha\xi_a/2)$.

The local impedance $Z_{loc}(x)$ (or LImp), defined as the quotient of the pressure $p(x)$ and the fluid velocity $v_x(x)$, becomes

$$Z_{loc}(x) = i\frac{J_0(z) + \gamma Y_0(z)}{J_1(z) + \gamma Y_1(z)}\left(\frac{h\rho C_0}{2}\right)^{1/2} \times \exp(-\alpha\xi/2). \quad (6)$$

For waves that transport energy to the right the real part of $Z_{loc}(x)$ will be positive. Since in the present case we will have mainly waves transporting energy to the left we will invert $Z_{loc}(x)$ so as

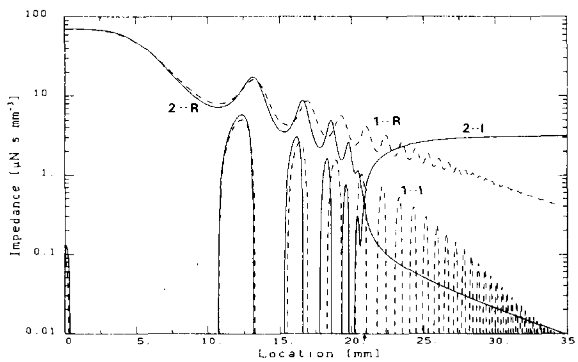

Fig. 1. Local impedance functions for 1000 Hz. (R) real part; (I) imaginary part. Curves: (1) model E; (2) model M. Abscissa: location x along length of BM (in mm). Arrow marks the resonance location for 1000 Hz in model M. Excitation at the helicotrema, termination at the stapes with the classical characteristic impedance.

to obtain a function of which the real part is again positive.

Before we discuss results for reverse waves it is convenient to recall the properties of the local impedance for forward waves. Fig. 2 of De Boer et al. (1985) shows the 'characteristic impedance' and the local impedance for a model driven from the stapes. Both are smooth and monotonic functions of x. The fact that the two functions differ betrays that a small but non-negligible degree of reflection occurs, this effect is larger near the stapes where the wavelength is larger and it is almost absent in the region of resonance. The properties of reverse waves are quite different. Fig. 1 shows the local impedance for reverse waves for 1000 Hz. The model was driven at the location corresponding to the helicotrema ($x = 35$ [mm]) and loaded by the characteristic impedance at the stapes ($x_a = 0$). Curves 1-R and 1-I refer to model E, curve 1-R showing the real part (as said, made positive) and curve 1-I the imaginary part (only positive lobes are shown, negative lobes suppressed). Curves 2-R and 2-I refer to model M (the 'complete' model). Parameter values are equal to those described in De Boer (1980), the value of δ was 0.05.

The pronounced undulations of the curves indicate that reverse waves undergo substantial reflection. Model E (curves 1) displays these reflections also in the abscissa range where model M has its resonance. This occurs despite the fact that everywhere to the left of this range model E has a larger BM resistance component than model M. Therefore, the fact that in model M the reflection of reverse waves in the region of resonance does not have a great influence on the local impedance is not caused by damping in the part of the cochlea that is stiffness-controlled. It must be a specific property associated with the resonance of the BM impedance $\zeta(x)$. Thus we see that it is necessary to attack directly the (somewhat more subtle) problem of forward and reverse waves in the resonance region.

Model R: BM impedance a linear function of x

To investigate what happens in the resonance region we use a simplified expression for the dependence of the BM impedance $\zeta(x)$ on x, namely, a linear one:

$$\zeta(x) = i\beta(x - x_0), \tag{7}$$

where β is given by

$$\beta = \alpha\omega M_0. \tag{8}$$

The point $x = x_0$ is the resonance location for the radian frequency ω of interest. The value of β is chosen in such a way that $\zeta(x)$ has the same slope as the reactance part of Eqn. 1a at this point. To account for losses, x is replaced by $\xi = x - x_0 - i\alpha/\delta$. In the present case this produces a constant resistance term in $\zeta(x)$. The general form of the solution for the pressure $p(x)$ is derived in De Boer and MacKay (1980) and reads:

$$p(x) = p_0 u\{J_1(u) + \gamma Y_1(u)\}, \tag{9}$$

where u is related to ξ as

$$u = 2i(2\xi\rho/\alpha h M_0)^{1/2}. \tag{10}$$

Here we drive the model from the right and prescribe that at the location $x = x_a$ (to which corresponds $u = u_a$) the model is loaded by the impedance Z_a. This condition leads to the following equation for γ:

$$\gamma = -\frac{u_a J_1(u_a) + Z_{ar} J_0(u_a)}{u_a Y_1(u_a) + Z_{ar} Y_0(u_a)}, \tag{11}$$

where Z_{ar} is a 'reduced' form of Z_a:

$$Z_{ar} = Z_a \frac{4}{i\omega M_0 \alpha h}. \tag{12}$$

In this case the local impedance is given by

$$Z_{loc}(x) = \frac{i\omega M_0 \alpha h u}{4} \times \frac{J_1(u) + \gamma Y_1(u)}{J_0(u) + \gamma Y_0(u)}, \tag{13}$$

and, as before, we plot it with inverted sign. Referring to De Boer and MacKay (1980) we must use the path of ξ that lies in the fourth quadrant (real part positive, imaginary part negative) when x passes through the resonance location.

We illustrate two cases. In one, models M and R are loaded at the stapes location $x = x_a = 0$ by their respective characteristic impedances, in the other one they are short-circuited at this point. Fig. 2 shows the former case. Curves 1-R and 1-I refer to model R, curves 2-R and 2-I to model M. As before, the damping coefficient δ is taken as equal to 0.05. The curves labelled 1 show little or no evidence of reverse waves being reflected, this is because model R is matched to its load at the stapes location and has a much smaller degree of inhomogeneity than model M. Hence, in the vicinity of the resonance point the local impedance should be nearly the same for forward and reflected waves, except for sign. The curves labelled 2 display the expected degree of undulation. Yet the course in the vicinity of the resonance point is very little affected by the reflections.

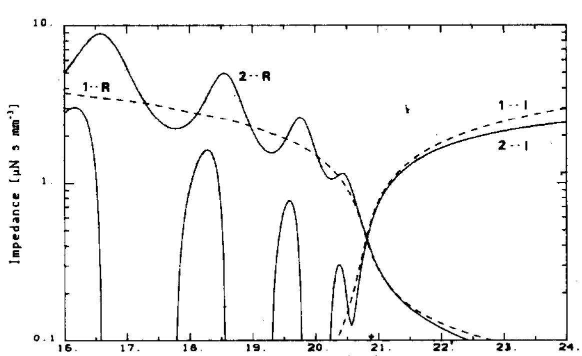

Fig. 2. Local impedance function for 1000 Hz. Curves: (1) model R; (2) model M. The abscissa scale is expanded to show more details around the resonance location (arrow). Excitation at the helicotrema, termination at the stapes with classical characteristic impedance. Note that in model R the reverse wave is not reflected at the stapes. Model M has larger and faster variations of the stiffness and produces noticeable reflection.

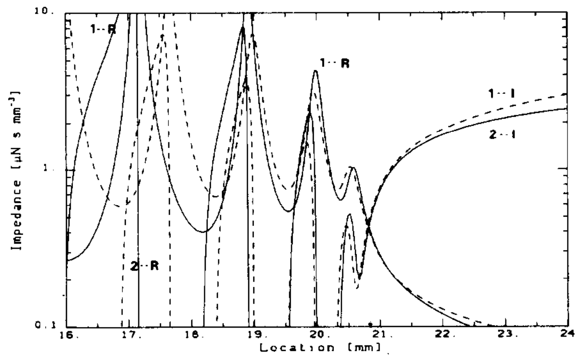

Fig. 3. As Fig. 2 but with the stapes terminated by a short circuit. The reflections are much more pronounced but the behaviour in the direct vicinity of the resonance location (arrow) is hardly affected.

Fig. 3 illustrates the case where the models are short-circuited at $x = x_a = 0$. In both models a substantial amount of reflection takes place, in the case of model R these reflections are nearly exclusively due to the mismatch at the stapes location. Again we see, however, that the impedance near the resonance point is only slightly affected by the reflection. We note that the simple expedient of replacing the reactance of $\zeta(x)$ by its tangent at $x = x_0$ is sufficient to demonstrate this property. The nature of this property must be contained in the solution Eqn. 9 with arbitrary γ. This leads us to yet another approach to the problem as detailed in the next section.

Energy flow considerations

Formulated in a different way, the problem at hand is as follows. Assume that a source injects vibratory energy at a certain location x_0 with a frequency that is nearly equal to the frequency for which x_0 is the resonance location. The source sets up waves to the right (forward waves) and to the left (reverse waves). The right-going wave will disappear after a few millimeters because of dissipation. The left-going wave can propagate all the way to the stapes but it may undergo substantial reflection. The so-reflected wave – a forward wave again – returns to the source but this appears to occur with such a reduced amplitude that it hardly affects the local impedance. Preceding sections show that the amplitude reduction is not due to damping in the stiffness-dominated part of the cochlea, nor is it due to interference of waves generated at different locations between stapes and resonance location. This is evident from the results of model E. The amplitude reduction is an intrinsic property associated with the course of the impedance in the direct vicinity of the resonance location (model R). It is not interference of forward waves in the latter region that causes the reduction since in that case wave energy would be reflected and it is known that forward waves are dissipated in the resonance region, and not reflected (De Boer and MacKay, 1980).

The last consideration leads us to the topic to be studied next: propagation and dissipation of energy, especially in the resonance region. We take model R and consider the forward wave. It is described by Eqn. 9 with $\gamma = +i$ (see De Boer and MacKay, 1980). From the considerations put forward in De Boer and MacKay (1980) it is easy to find that the power flow density W_1 per unit of channel cross-sectional area is given by

$$W_1 = \tfrac{1}{2}\mathrm{Re}[\,p(x)v_x^*(x)]$$
$$= \frac{p_0^2}{\omega\rho}\mathrm{Im}[\,u\{J_1(u) + iY_1(u)\}$$
$$\times\{-J_0^*(u) + iY_0^*(u)\}], \qquad (14)$$

where Re [.] and Im [.] stand for real and imaginary parts of the argument, respectively, and the asterisk denotes the complex conjugate. The quantity W_1 is depicted in Fig. 4a as a function of x, again for the typical case where the damping coefficient δ equals 0.05. The ordinate is labelled in dB and is normalized so that the curve depicted starts at 0 dB at the left margin of the figure. As before, the operating frequency is 1 000 Hz. At the resonance location, marked by an arrow, the power

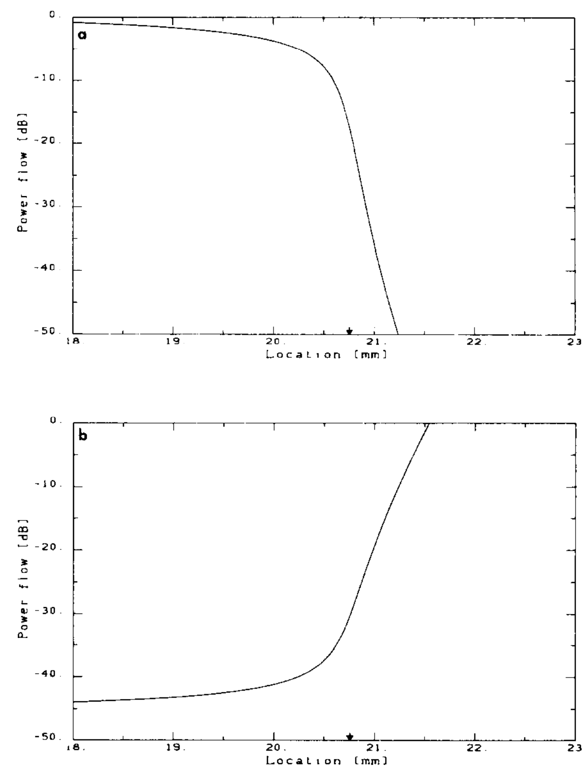

Fig. 4 (a) Power flow density in model R. Excitation at the stapes. At the resonance location (arrow) the power is observed to be reduced by approx. 18 dB. (b) Power flow density in model R. Excitation at the helicotrema. Quite arbitrarily, the level is made to lie at -30 dB at the resonance location (arrow). Variations around that point are approximately opposite to those shown by (a).

loss is observed to be approximately 18 dB. For the explanation of this finding we refer to De Boer (1980) where the power flow is (under)estimated on the basis of the WKB (or LG) approximation.

For the sake of completeness we present in Fig. 4b the result for a reverse wave. The power flow density W_1 can be found from Eqn. 14 with i replaced by γ and γ computed from Eqn. 11. To simplify matters we take the case corresponding to Fig. 2 where model R is driven from the helicotrema and loaded at the stapes by its characteristic impedance. On its path from helicotrema to resonance location the injected wave undergoes, of course, a tremendous amount of attenuation. Only a small part of it is shown in the figure which concentrates on what happens in the resonance region. In many respects the energy loss per mm in the resonance region is similar to that in Fig. 4a so that the two curves are almost complementary to one another. Detailed analysis shows the attenuation of reverse waves to be consistently somewhat smaller than that for forward waves. The reason for this (probably unimportant) effect has not yet been found.

On the basis of these findings we can draw the following inferences. Assume a source of acoustic energy is located at or near the resonance location for the frequency under consideration. The component of the injected wave that travels to the left will undergo an attenuation of the order of 18 dB (or more) on its path toward the stapes, most of this attenuation occurs in the direct vicinity of the resonance point. If that wave is reflected at the stapes without further loss it will undergo an additional attenuation of the same magnitude before it reaches the source again. On its 'round trip' from the source and back to it again the wave thus suffers an attenuation of at least 35 dB. This is the reason why in the resonance region the local impedance for reverse waves depends so little on the acoustical conditions prevailing at the stapes. Moreover, the operating conditions for a source of oto-acoustic emission (for instance, the condition for stability) will depend very little on the load impedance at the stapes location (cf. De Boer et al., 1985). Note, furthermore, that at locations a short distance to the left of the resonance region the attenuation is much smaller so that large effects of the stapes impedance will be found there.

Discussion

The analyses presented thus far have shown and illuminated several things. The asymmetry with respect to direction of wave travel, for instance, is found already in the most simplified model, the exponential model (E). Let us consider what happens to forward and reverse waves in different regions of that model. At the extreme right-hand part the wavelength is very small compared to $1/\alpha$, the inverse of the space constant, and the variable z in Eqn. 1 is very large. So large, in fact, that asymptotic expressions can be used for the Bessel functions. The resulting expression for the pressure $p(x)$ then represents a simple linear combination of a wave travelling to the left and one travelling to the right. Their relative magnitudes would depend, for instance, on a boundary condition elsewhere. The principal fact is that the two waves propagate independently and do not interact. In other words, neither of the two waves undergoes intrinsic reflection despite the inhomogeneity of the model. At the other extreme we have the case where the wavelength is large compared to $1/\alpha$. This will occur near the stapes (at least for not too high frequencies). In this case the two terms of Eqn. 1a do not allow for a simple physical interpretation. Each term implies a forward and a reverse wave. In other words, each type of wave cannot exist without the other, any wave will undergo appreciable reflection. There remains the intermediate case where the magnitude of z is of the order of unity. The forward wave, described by Eqn. 2 with $\gamma = -i$, will suffer little reflection as is proved in De Boer and MacKay (1980). Every other combination of the two terms represents a wave that undergoes some non-negligible reflection. This holds especially for waves that are set up as reverse waves, e.g. by exciting the model at a place to the right of the resonance location.

The second point proved in the preceding sections concerns the peculiar property that operating conditions for a source of oto-acoustic emission depend so little upon reflection at the stapes. This property is found to be due to the large sound power attenuation that occurs over the resonance region. In short, the attenuation of a sound wave injected in the resonance region, for a round trip from source to stapes and back again, is so large (more than 35 dB) that the returning wave is too weak to have any influence on the source.

The situation would be entirely different, of course, in an active model. Suppose conditions are such that some 18 dB of amplification of the forward wave occurs due to activity of the BM. The same amplification should occur for reverse waves, of course. This would compensate for the loss due to dissipation (up to the resonance point). The reflected reverse wave would tend to make a potential source located at the resonance point unstable and the degree of reflection occurring at the stapes would influence this very much. In particular, the stability of any part of the active BM in the resonance region would become strongly dependent upon the (phase of the) reflection occurring at the stapes. This would set an upper limit to the amount of sound amplification that could be built into an active model. Actually, this limit is very much on the conservative side since stable active models have been constructed with over 40 dB of amplification. Several other factors apparently contribute to stability. Or, the analysis in terms of a single source is too crude. Nevertheless, the reasoning presented here gives a different outlook on the question of stability (which is the central problem in the realm of active models).

One very obvious limitation of the approach presented here is that it is limited to long waves. Analytical treatment of forward and reverse waves in a short-wave model is a somewhat intricate affair (cf. De Boer, 1983). Work in this field is in progress and results will be published later. Numerical calculations on complex models supporting long as well as short waves can provide additional information, of course. The main problem in constructing such models is that a detailed account should be given of the way activity arises. That is, the dynamics of the process that produces active behaviour should be described in such quantitative detail that it is clear why active effects are observed only in the region up to the response peak, and nowhere else. Although this has been done to a considerable extent (cf. Neely and Kim, 1983) it remains difficult to extract fundamental physical features from results of computations in such models. To obtain real insight, it is mandatory to interpret such numerical results with reference to results obtained from models that are sufficiently simplified to allow an analytical solution. As mentioned, further analytical work is presently undertaken and much will depend on the confrontation of both types of results.

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